Technical Comments

Comment on "Distributed Mass Matrix for Plate Element Bending"

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THE displacement functions for a rectangular plate element in bending, using the polynomial terms that obey the biharmonic equation, have now been published five times.[1-5] In view of the concluding remarks,[4] the plate considered evidently has uniform thickness, so that the mass matrix can be found explicitly. This has been done[5,7] and checked by the author, who added one instruction to a program intended for a quite different use. May one enquire whether the publication of such results is generally beneficial?

It is surprising to find explicit energy matrices in learned American journals when a large sector of the British airframe industry already has some such tedious work automated. The author proposed numerical integration independently some four years ago and now favors a technique that divorces from the main program the calculation of displacement functions and their derivatives at an integrating point, by writing it as a subroutine.

From this subroutine a rectangular matrix $[L]$ is assembled which operates on the nodal deflections to give the strains $\epsilon = Lx$, so that the strain energy is

$$\tfrac{1}{2}x'\textstyle\int L'DLx.d\text{ (volume)} = \tfrac{1}{2}x'[\Sigma kJL'DL]x \qquad (1)$$

where k is the integrating constant resulting from multiple applications of a Gauss rule of integration, and J is a Jacobian; nontrivial if dimensionless coordinates are used in a quadrilateral, a triangle, or an element with a curved edge, for example. Kinetic energy, or second-order strain energy for stability calculations, is treated with little effort, as are different elastic problems that happen to use the same element geometry. Having found the nodal deflections, one can enter the subroutine again to find stresses, deflections between nodes, etc.

Thus, the programing effort, debugging costs, and checking are reduced, and the scope for experiment appears unlimited. In a simple case like that under discussion, the correct answer is found with few integrating terms, but as displacement functions become more complicated the choice of integrating accuracy must become an economic question.

Calculation of (1) is most cheaply accomplished[6] as follows:

$$\begin{bmatrix} -k^{-1}D^{-1} & L \\ L' & O \end{bmatrix} \begin{Bmatrix} \sigma \\ x \end{Bmatrix} \qquad (2)$$

where the matrix is symmetrically reduced as if notionally eliminating σ from a set of equations. Accumulation is automatic, or if extreme economy of storage is desirable one can accumulate directly into the assembled stiffness matrix. In any case, numerical integration techniques use very little storage. It is advisable to use the largest k last, to reduce roundoff error.

The author feels strongly that the research effort should be diverted to basic questions, such as the choice of displacement functions, the understanding of roundoff errors, etc.

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References

1 Adini, A., "Analysis of shell structure by the finite element method," Ph.D. Thesis, Univ. of California (June 1961).
2 Zienkiewicz, O. C. and Chueng, Y. K., "The finite element method of analysis for arch dam shells and comparison with finite difference procedures," *Theory of Arch Dams* (Pergamon Press Ltd., London, 1965).
3 Melosh, R. J., "Basis for derivation of matrices for the direct stiffness method," AIAA J. 1, 1790–1795 (1963).
4 Guyan, R. J., "Distributed mass matrix for plate element bending," AIAA J. 2, 567–568 (1965).
5 Dawe, D. J., "A finite element approach to plate vibration problems," J. Mech. Eng. Sci. 7, 29–32 (1965).
6 Faddeeva, V. N., *Computational Methods of Linear Algebra* (Dover Publications, Inc., New York, 1959), p. 90.
7 Zienkiewicz, O. C. and Cheng, Y. K., "The finite element method for analysis of elastic isotopic and orthotropic slabs," Proc. Inst. Civ. Eng. London 28, 471–488 (1964).

Comment on "Magnetohydrodynamic Flow Past a Thin Airfoil"

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IN a previous paper[1] the Riemann invariants derived from the linearized system of Lunquist equations for general two-dimensional magnetohydrodynamic flow are given. Unfortunately, the solution for the invariants associated with the quadratic factor (in terms of the original notations of Ref. 1)

$$(\lambda \sin\theta + \cos\theta)^2 = M_0^2 \qquad (1)$$

which hold along transverse characteristics and exist only in general two-dimensional magnetohydrodynamic flow that $b \neq 0$ and $B_0 \neq 0$ [c.f. (11) of Ref. 1] is in error. The correct expression should read

$$\mathbf{r}_\pm = [1, 0, \pm(1/M_0), \mp(\lambda_\pm/M_0), (b/M_0^2), 0] + J_\pm \times [0, -\csc\theta, 0, 0, \pm(1/bM_0), 1/b] \qquad (2)$$

where

$$\lambda_\pm = (\pm M_0 - \cos\theta)\csc\theta \qquad (3)$$

as solved from (1) and

$$J_\pm = \cos\theta \pm M_0\{[\tan^2\theta(1 + \lambda_\pm^2)/(1 + \lambda_\pm \tan\theta)^2] - 1\} \qquad (4a)$$

which, by the use of (3), simplifies to

$$J_\pm = \pm(1/M_0) - \cos\theta \qquad (4b)$$

Substitution of $\lambda_\pm$ in (3) and $J_\pm$ in (4b) into (2) finally

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